BDDC and FETI–DP algorithms for mortar finite element methods

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Outline

1. Mortar discretization
2. BDDC and FETI–DP algorithms
3. Extensions
4. Numerical results
5. Conclusion

Mortar Discretization

- **A model elliptic problem**

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

- **Finite element space** of **nonmatching** triangulation

Figure 1: Matching grids(left) and nonmatching grids(right)

  – Adaptivity: singular points, discontinuous coefficients

  – Mesh generation, Multi-physics simulation

  – Nonconforming $\not\subseteq H^1(\Omega)$

- **Mortar matching condition**

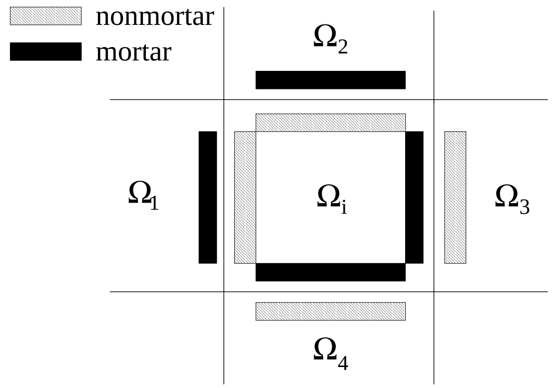

Figure 2: mortar, nonmortar sides

1. decide nonmortar, mortar
2. build Lagrange multiplier space M_{ij} based on the nonmortar, ($1 \in M_{ij}$, d.o.fs)
3. mortar matching condition

$$\int_{\Gamma_{ij}} (v_i - v_j)\lambda \, ds = 0 \quad \forall \lambda \in M_{ij}. \tag{2}$$

- **Mortar discretization** is to approximate the solution u in the finite element space satisfying **the mortar matching condition**.

Matrix Representation of Mortar Discretization

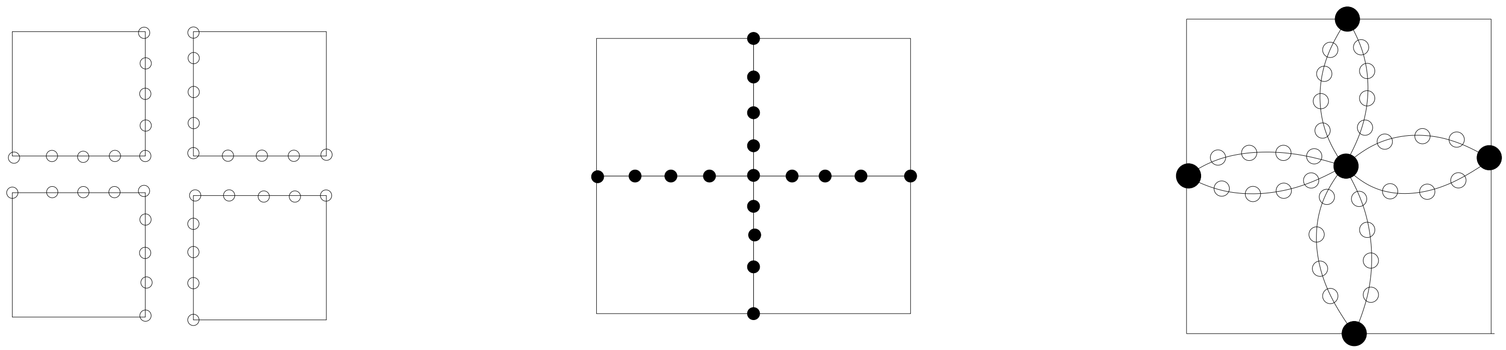

Figure 3: spaces $W = \prod_{i=1}^{N} W_i$ (left), $\widehat{W}$ (center), $\widetilde{W}$ (right)

- **Finite element Spaces**

$$X_i \; : \; \text{finite element space in } \Omega_i, \quad W_i \; : \; \text{trace space of } X_i, \quad X_i(\partial\Omega_i),$$

$$W = \prod_{i=1}^{N} W_i \quad \textbf{discontinuous} \text{ across the interface},$$

$$\widehat{W} \subset W \quad \text{with } \textbf{mortar matching condition} \text{ ,}$$

$$\widetilde{W} \subset W \quad \text{with } \textbf{primal constraints}.$$

Matrix Representation of Mortar Discretization (cont.)

- **Primal constraints**

  The primal constraints are selected from **the mortar matching condition** so that $\widehat{W} \subset \widetilde{W}$;

  For example, **continuity at corners** or **averages on edges/faces**.

  In the following, we simply consider **continuity at corners** as primal constraints.

Matrix Representation of Mortar Discretization (cont.)

- Equations of the mortar matching condition on $\widetilde{W}$

  Separate unknowns $w \in \widetilde{W}$ into

$$w_n \text{ (nonmortar) }, \quad w_m \text{ (mortar) }, \quad w_c \text{ (corners)}.$$

The Eqns. of the mortar matching condition is

$$B_n w_n + B_m w_m + B_c w_c = 0, \tag{3}$$

$$w_n = -B_n^{-1}(B_m w_m + B_c w_c).$$

- The space W_G of unknowns (w_m, w_c).
- Mortar finite element space $\widehat{W}$ is represented by W_G,

$$\begin{pmatrix} w_n \\ w_m \\ w_c \end{pmatrix} = R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = \begin{pmatrix} -B_n^{-1}B_m & -B_n^{-1}B_c \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w_m \\ w_c \end{pmatrix}, \quad \forall (w_m, w_c) \in W_G.$$

Matrix Representation of Mortar Discretization (cont.)

- Local Schur complement matrix

$$S^{(i)} = \begin{pmatrix} S_{rr}^{(i)} & (S_{cr}^{(i)})^t \\ S_{rc}^{(i)} & S_{cc}^{(i)} \end{pmatrix}, \quad c\text{: corners, } r\text{: remaining.} \tag{4}$$

- Subassembly of $S^{(i)}$, (gluing unknowns at corners)

$$\widetilde{S} = \begin{pmatrix} S_{rr} & S_{cr}^t \\ S_{cr} & S_{cc} \end{pmatrix}, \tag{5}$$

$$S_{rr} = \mathrm{diag}(S_{rr}^{(i)}),$$

$$S_{cr} = \begin{pmatrix} (R_c^{(1)})^t S_{cr}^{(1)} & \cdots & (R_c^{(N)})^t S_{cr}^{(N)} \end{pmatrix},$$

$$S_{cc} = \sum_{i=1}^{N} (R_c^{(i)})^t S_{cc}^{(i)} R_c^{(i)},$$

$R_c^{(i)}$ restriction of the primal unknowns to subdomain Ω_i.

Matrix Representation of Mortar Discretization (cont.)

- **The Eqns. of the mortar discretization is**

$$R\widetilde{S}R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix}, \quad R^t = \begin{pmatrix} -B_n^{-1}B_m & -B_n^{-1}B_c \\ I & 0 \\ 0 & I \end{pmatrix}. \tag{6}$$

- **Equivalent dual problem**

The above problem (6) is equivalent to

$$\max_{\lambda} \min_{w \in \widetilde{W}} \left\{ \frac{1}{2} \langle \widetilde{S}w, w \rangle + \langle \widetilde{g}, w \rangle + \langle Bw, \lambda \rangle \right\}, \tag{7}$$

where

$$B = \begin{pmatrix} B_n & B_m & B_c \end{pmatrix}.$$

After eliminating unknowns other than λ, the equations on dual variables λ follow,

$$B\widetilde{S}^{-1}B^t\lambda = B\widetilde{S}^{-1}\widetilde{g}, \tag{8}$$

BDDC and FETI–DP algorithms

- **BDDC algorithm**

$$R\widetilde{S}R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix}, \text{ using } RD\widetilde{S}^{-1}DR^t. \tag{9}$$

- **FETI–DP algorithm**

$$B\widetilde{S}^{-1}B^t\lambda = B\widetilde{S}^{-1}\widetilde{g}, \text{ using } B\Sigma\widetilde{S}\Sigma B^t. \tag{10}$$

- The weights D and Σ are chosen to give the optimal condition number bound,

$$\kappa(B_{DDC}),\ \kappa(F_{DP}) \simeq C(1+\log\frac{H}{h})^2, \quad \frac{H}{h} : \text{ the size of the local problem}$$

$$B_{DDC} = (RD\widetilde{S}^{-1}DR^t)R\widetilde{S}R^t, \quad F_{DP} = (B\Sigma\widetilde{S}\Sigma B^t)B\widetilde{S}^{-1}B^t.$$

BDDC and FETI–DP algorithms (cont.)

- **FETI-DP algorithm for mortar (By Kim and Lee)**

The FETI-DP algorithm solves iteratively

$$B\widetilde{S}^{-1}B^t\lambda = B\widetilde{S}^{-1}\widetilde{g}, \tag{11}$$

with **the Neumann-Dirichlet preconditioner**

$$B\Sigma\widetilde{S}\Sigma B^t, \tag{12}$$

of **the weights**,

$$\Sigma = \begin{pmatrix} \Sigma_{nn} & 0 & \\ 0 & \Sigma_{mm} & 0 \\ 0 & 0 & \Sigma_{cc} \end{pmatrix}, \quad \Sigma_{nn} = (B_n^t B_n)^{-1},\ \Sigma_{mm} = 0,\ \Sigma_{cc} = 0.$$

The optimal condition number bound has been shown,

$$\kappa(F_{DP}) \simeq C\,(1 + \log(H/h))^2\,(\text{indep. of coefficient jumps}). \tag{13}$$

BDDC and FETI–DP algorithms (cont.)

- **BDDC algorithm for mortar**

  We solve for the primal unknowns (w_m, w_c) iteratively

$$R\widetilde{S}R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix} \tag{14}$$

  using a preconditioner of the form

$$RD\widetilde{S}^{-1}DR^t. \tag{15}$$

  Our aim is to find weights D for the BDDC algorithm that has the same spectra as the FETI-DP algorithm.

  So that we obtain **the optimal converge** as well

$$\kappa(B_{DDC}) \simeq C\,(1 + \log(H/h))^2\,.$$

BDDC and FETI–DP algorithms (cont.)

- **Connection between BDDC and FETI-DP for mortar**

  We recall two algorithms

$$F_{DP} = (B\Sigma\widetilde{S}\Sigma B^t)B\widetilde{S}^{-1}B^t, \quad B_{DDC} = (RD\widetilde{S}^{-1}DR^t)R\widetilde{S}R^t.$$

  We define

$$P_\Sigma = \Sigma B^t B \text{ (jump)}, \quad E_D = R^t RD \text{ (average)}. \qquad (16)$$

  Theorem 1 (Li and Widlund) *If they satisfy*

$$P_\Sigma + E_D = I$$
$$E_D^2 = E_D, \quad P_\Sigma^2 = P_\Sigma,$$
$$E_D P_\Sigma = P_\Sigma E_D = 0,$$

  then the operators B_{DDC} and F_{DP} have the same spectra except the eigenvalue 1.

BDDC and FETI–DP algorithms (cont.)

The weights

$$D = \begin{pmatrix} D_{nn} & & \\ & D_{mm} & \\ & & D_{cc} \end{pmatrix}, \quad D_{nn} = 0,\ D_{mm} = I,\ D_{cc} = I$$

satisfy the above properties when Σ is given by

$$\Sigma = \begin{pmatrix} \Sigma_{nn} & 0 & \\ 0 & \Sigma_{mm} & 0 \\ 0 & 0 & \Sigma_{cc} \end{pmatrix}, \quad \Sigma_{nn} = (B_n^t B_n)^{-1},\ \Sigma_{mm} = 0,\ \Sigma_{cc} = 0.$$

Note: The pair (Σ, D) is unique that satisfy the above properties with the optimal condition number bound.

Extensions

▷ Primal constraints other than continuity at corners

3D elliptic problems, 3D elasticity problems
(need richer primal constraints than corners)

For example,

$$\int_{F_{ij}} (v_i - v_j)\, ds = 0, \quad (\text{Note : } 1 \in M_{ij} \ ),$$

$$\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot P_{ij}\mathbf{r}\, ds = 0, \quad (\ \mathbf{r} : \text{ rigid body motions }),$$

P_{ij} : proj. onto Lagrange multiplier space.

Extensions (cont.)

▷ Geometrically nonconforming partition.

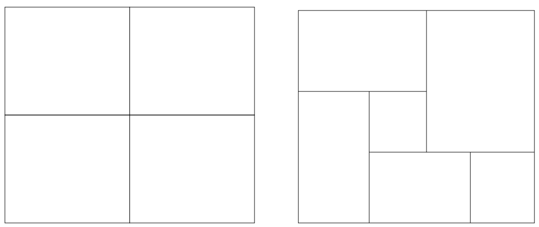

Figure 4: Geometrically conforming (left) and geometrically non-conforming partitions (right).

A nonmortar $F \subset \partial\Omega_i$ partitioned by its mortar neighbors Ω_j,

$$F = \cup_j F_{ij}, \quad F_{ij} = \partial\Omega_i \cap \partial\Omega_j,$$

a function ϕ from its mortar neighbors by

$$\phi = w_j \text{ on } F_{ij}. \tag{17}$$

Mortar matching condition is

$$\int_F (w_i - \phi)\,\psi\,ds = 0, \quad \forall \psi \in M(F). \tag{18}$$

$M(F)$: Lagrange multiplier space

Extensions (cont.)

- Primal constraints depending on each F_{ij}

$$\int_{F_{ij}} (w_i - w_j)\psi_{ij}\, ds = 0,$$

ψ_{ij} is **the sum of Lagrange multipliers bases** supported in F_{ij}.

- Primal constraints depending on F (elasticity)

$$\int_F w_i\, ds = \int_F \phi\, ds,$$

$$\frac{1}{|F|}\int_F w_i\, ds = \sum_j a_{ij} \frac{1}{|F_{ij}|}\int_{F_{ij}} w_j\, ds, \quad a_{ij} = \frac{|F_{ij}|}{|F|},$$

$$\overline{w_i}^F = \sum_j a_{ij} \overline{w_j}^{F_{ij}}. \tag{19}$$

- Slightly weaker condition number bound $(1 + \log(H/h))^3$.

Numerical Results

- **Comparison of BDDC and FETI-DP algorithms**

  Model problem

$$-\Delta u(x,y) = f(x,y) \text{ in } \Omega = [0,1] \times [0,1]$$
$$u(x,y) = 0 \text{ on } \partial\Omega$$

  Exact solution: $u(x,y) = y(1-y)\sin \pi x$

  CGM: stopping criterion $\rightarrow$ relative residual $\leq$ 1.0e-6

  N: the number of subdomains

  n: the number of nodes on the subdomain edge including end points

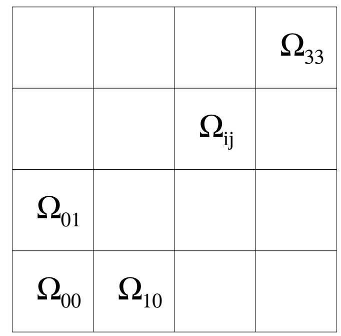

Figure 5: Partition of subdomains when $N = 4 \times 4$

Figure 6: Matching grids(left) and nonmatching grids(right) when $n = 5$

Table 1: (Non-matching grids) Comparison of FETI–DP and BDDC algorithms

$N = 4 \times 4$	F_{DP}		B_{DDC}		$n = 5$	F_{DP}		B_{DDC}	
$n-1$	$\lambda_{\min}$	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}$	N	$\lambda_{\min}$	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}$
4	1.40	4.09	1.00	4.09	4×4	1.40	4.09	1.00	4.09
8	1.01	5.72	1.00	5.72	8×8	1.37	4.41	1.00	4.41
16	1.00	7.72	1.00	7.72	16×16	1.32	4.49	1.00	4.49
32	1.01	1.00e+1	1.00	1.00e+1	32×32	1.30	4.57	1.00	4.62
64	1.01	1.28e+1	1.00	1.28e+1					

Conclusion

1. FETI-DP with the Neumann-Dirichlet preconditioner
   - ▷ Elliptic problems in $2D$, $3D$
   - ▷ Stokes problem in $2D$
   - ▷ $3D$ compressible elasticity
   - ▷ The most efficient for the problems with coefficient jumps
2. A BDDC algorithm well connected to FETI-DP with ND-preconditioner
   - ▷ Extended to elliptic problems in both $2D$ and $3D$
   - ▷ $3D$ compressible elasticity
   - ▷ Geometrically nonconforming subdomain partitions.